J. Fluid Mech. (2008), vol. 598, pp. 227–244. © 2008 Cambridge University Press
doi:10.1017/S0022112007009974 Printed in the United Kingdom

The initial-value problem for three-dimensional disturbances in plane Poiseuille flow of helium II

LARS B. BERGSTRÖM
Department of Mathematics, Luleå University of Technology, SE-971 87 Luleå, Sweden

(Received 5 October 2006 and in revised form 5 November 2007)

The time development of small three-dimensional disturbances in plane Poiseuille flow of helium II is considered. The study is conducted by considering the interaction of a normal fluid field and a superfluid field. The interaction is caused by a mutual friction forcing between the two flow fields. Specifically, the stability of the normal fluid affected by the mutual forcing is considered. Compared to the ordinary fluid case where the mutual forcing is not present, the presence of the mutual forcing implies a substantial increase of the transient growth of the disturbances. The increase of the transient growth occurs because the mutual forcing reduces the damping of the disturbances. The phase of transient growth becomes thereby more prolonged and higher levels of amplification are reached. There is also a minor effect on the transient growth caused by the modification of the mean flow owing to the mutual forcing. The strongest transient growth occurs for streamwise elongated disturbances, i.e. disturbances with streamwise wavenumber $\alpha = 0$. When α increases beyond zero, the transient amplification quickly becomes reduced. Striking differences compared to the ordinary fluid case are that the largest transient amplification does not occur when the spanwise wavenumber (β) is close to two and that the peak level of the disturbance energy density amplification does not depend on the square of the Reynolds number.

1. Introduction

Hydrodynamic stability of parallel shear flows constitutes a fundamental and, to a great extent, unsolved problem in science and engineering. A frequently investigated case of hydrodynamic stability is the stability problem of plane Poiseuille flow. In early theoretical investigations by Heisenberg (1924), Tollmien (1929), Schlichting (1933) and Lin (1945), results derived by using heuristic and asymptotic methods indicated that the Reynolds number (R) above which small disturbances may grow exponentially is 5000. For a long period of time most theoretical studies of the stability were focused on the two-dimensional (i.e. disturbances without a spanwise dependence) eigenvalue problem. An important reason for that is that for every unstable three-dimensional disturbance, there is a two-dimensional disturbance unstable at a lower Reynolds number. For classical stability, it is therefore only necessary to consider the problem of two-dimensional disturbances. This was originally proved by Squire (1933) who introduced the well-known transformation that reduces the three-dimensional problem to a two-dimensional problem. Although studies of the linearized two-dimensional problem gave a critical Reynolds number well above 5000, experiments by Davies & White (1928) had shown that transition to turbulence in plane Poiseuille flow can occur at values of R as low as 1000. The discrepancy between the findings of linear theory and experiments was at that time ascribed to nonlinear effects. In weakly nonlinear theory, Meskyn & Stuart (1951) applied the so-called mean-field theory and found that finite-amplitude steady waves exist for Reynolds numbers substantially lower than 5000. A nonlinear critical Reynolds number of about 2900 was suggested. By expanding the equations governing the disturbances in powers of the disturbance amplitude, Stuart (1960) and Watson (1960) derived Landau type equations for the nonlinear development of the disturbances. The Landau coefficients then determine the nonlinear stability characteristics of the disturbances. Landau coefficients were subsequently calculated by Reynolds & Potter (1967), Pekeris & Scholler (1971) and Herbert (1976). For higher-order approximations, Herbert (1976) found a nonlinear critical Reynolds number of 2935. This was substantially lower than the 'exact' linear critical Reynolds number of 5772 established numerically by Orszag (1971). Since experimental investigations of plane Poiseuille flow indicated that transition may occur for R around 1000 (Davies & White 1928), although nonlinearity reduces the linear critical Reynolds-number, the nonlinear critical Reynolds number found was still well above the experimentally obtained Reynolds-number limit for transition. In some other types of parallel shear flow, there was an even more striking lack of coherence between theoretical and experimental findings. For example, theoretical studies of pipe Poiseuille flow and plane Couette flow showed that neutral stable curves do not even exist.

The discrepancy between theoretical and experimental results of instability in all types of parallel shear flows directed ideas towards the possibility of other transition mechanisms that would bypass the traditional concept of exponentially unstable disturbances. The idea of other routes to transition was first introduced by Morkovin (1969) who suggested that it might be possible to bypass exponential growth of disturbances if the exponential growth could be replaced with some other strongly amplifying mechanism. One such mechanism is the so-called non-modal mechanism on which work on the hydrodynamic stability of parallel shear flows has been focused during the last few decades. In studies of the linearized two-dimensional initial-value problem, Farrell (1988) found that damped disturbances in plane Poiseuille flow could undergo a rapid phase of amplification, a so-called transient amplification. The disturbance energy density amplification was about twenty-fold for $R = 1000$, i.e. the lowest R for transition found in experiments. In the case of small three-dimensional disturbances, Gustavsson (1991) showed a substantially stronger amplification. The peak of the energy density amplification was nearly 180 for $R = 1000$ and occurred for streamwise independent structures. Butler & Farrell (1992) used variational techniques to calculate the largest possible transient amplification of three-dimensional disturbances, the so-called optimal disturbance amplification. The optimal amplification was found to be 196 for $R = 1000$ and occurred for streamwise independent disturbances with a spanwise wavenumber close to two. Also in plane Couette flow and boundary-layer flow, strong amplifications were reported by Butler & Farrell (1992). For example, in the plane-Couette-flow case, a disturbance energy density amplification of 1166 (for $R = 1000$) was found to occur owing to non-normality. Extensive results for the transient growth of disturbances in shear flows were also presented by Reddy & Henningson (1993). In the pipe-Poiseuille-flow case, Bergström (1993) calculated the optimal disturbance amplification for angular dependent disturbances. The results showed that the largest amplification occurs for streamwise elongated structures with an azimuthal wavenumber equal to one. For $R = 2000$, i.e. the lowest R for transition found in pipe-flow experiments, the optimal amplification of the disturbance energy density was 288. Similar results were reported by Schmid & Henningson (1994).

The possibility of a transient disturbance development has also been verified in a number of experimental investigations. In the plane-Poiseuille-flow case, by inducing a point-like initial disturbance, Klingmann (1992) found a strong subsequent disturbance amplification followed by decay. At high enough initial disturbance amplitude, the disturbance continued to grow and eventually transition occurred. In pipe Poiseuille flow, Bergström (1995) found experimental evidence for a transient disturbance development. By re-exploring data from Kaskel's (1961) pipe-flow experiment, Mayer & Reshotko (1997) also found evidence of a transient disturbance amplification in pipe Poiseuille flow.

The theoretical explanation for the observed strong amplification of damped three-dimensional disturbances is the non-normality of the operators governing the disturbances in all parallel shear flows. The eigenfunctions of non-normal operators are not orthogonal. An intrinsic property of non-orthogonal eigenfunctions is that a superposition of individually damped eigenfunctions causes a strong transient amplification of the disturbance represented by the eigenfunctions. Another way of explaining the strong amplification of three-dimensional disturbances is that the mean flow shear together with the normal velocity disturbance act as a forcing of the normal vorticity disturbance. This forcing is only present for three-dimensional disturbances. Physically, this process can be seen as the lifting up of fluid elements in the presence of a mean shear. The lift-up concept was introduced by Landahl (1975, 1980) who showed that the lifting up of fluid elements in the wall normal direction generates a streamwise disturbance velocity.

Although the transient growth of ordinary parallel shear flows has been thoroughly studied, there are fluids for which the transient growth of disturbances has not been investigated. One such fluid of particular interest is the superfluid helium II which has been found to exhibit a high-Reynolds-number turbulent flow similar to classical turbulence (see Smith *et al.* 1993; Barenghi *et al.* 1997). Helium II can be modelled as a superposition of a normal fluid and a superfluid where the superfluid has zero viscosity. The normal fluid and the superfluid interact via a mutual friction force localized at the position of the vortex lines of the superfluid. The mutual forcing then modifies the normal fluid flow field. The general form of the formula for the mutual forcing, $\boldsymbol{F}_{mutual}$, can be written (cf. Melotte & Barenghi 1998; Godfrey, Samuels & Barenghi 2001)

$$\boldsymbol{F}_{mutual} = \frac{B\rho_s L}{(\rho_s + \rho_n)V_o}\boldsymbol{\omega}_a \times [\boldsymbol{\omega}_a \times (\boldsymbol{U}_n - \boldsymbol{U}_s)], \tag{1.1}$$

where $\boldsymbol{\omega}_a$ is the vorticity of the superfluid averaged over many quantized vortex lines, $\boldsymbol{U}_n$ is the normal fluid velocity and $\boldsymbol{U}_s$ is the superfluid velocity. B represents the coefficient of the mutual friction and L and V_o are length and velocity scales, respectively. ρ_s and ρ_n are the density of the superfluid and the normal fluid, respectively. In (1.1), the ratio between the superfluid density and the total density $\rho_s/(\rho_s + \rho_n)$ is strongly temperature dependent. In §2, a model for the mutual forcing represented by (1.1) will be derived. In the model to be derived, the ratio $B\rho_s L/(\rho_s + \rho_n)V_o$ is represented by a constant which consequently will also be strongly temperature dependent. Since $\boldsymbol{\omega}_a$ in (1.1) represents the vorticity of the superfluid averaged over many quantized vortex lines, variations in the superfluid field may not have a dramatic influence on the mutual forcing term even if the superfluid field is not preserved. Studies of the hydrodynamic stability of helium II have usually taken a kinematic approach where the problem has been solved for one of the two fluid components. The other component has then been held fixed and acts as a forcing term. The problem of keeping the superfluid fixed and solving for the normal fluid was first done by Melotte & Barenghi (1998). They found a transition behaviour of the normal fluid in agreement with experimental findings.

By keeping the superfluid fixed, in investigations of the two-dimensional eigenvalue problem of plane Poiseuille flow of helium II, Godfrey *et al.* (2001) found two unstable branches, an upper branch and a lower branch. The upper branch corresponds to the instability branch of ordinary Poiseuille flow slightly shifted by the mutual friction forcing. The lower new branch, which does not exist in ordinary Poiseuille flow, has critical Reynolds numbers substantially lower than for the upper branch. At the lower branch, the critical Reynolds number was found to decrease when the strength of the mutual forcing increases. Since the classical stability characteristics (i.e. the question of whether the eigenfunctions are damped or not) of plane Poiseuille flow are modified by the mutual friction forcing, it is of interest to investigate how the mutual friction forcing will affect the transient growth properties of three-dimensional disturbances.

The main objective of the present paper is therefore to investigate the stability of the normal fluid field of helium II for transient growth. Specifically, the three-dimensional initial-value problem of small disturbances in plane Poiseuille flow of helium II will be addressed. It must, however, be emphasized that the physics of helium II is very complex and that the model to be derived to represent the superfluid interaction is a very simplified model. The model is a first approach to studying how the transient growth properties, that have been a focus in hydrodynamic stability for the last decade, will change by the presence of a forcing term representing the interaction with the superfluid. The model is not intended to describe the full transition to turbulence of helium II, but merely the initial growth of small disturbances in a laminar mean flow. In § 2, the equation governing the mean flow of the normal fluid affected by the mutual forcing is presented and solved. In § 3, the governing equations of the three-dimensional disturbance field are presented. In § 4, the system of equations is solved and results for the disturbance development are presented for different combinations of parameters. Finally, in § 5, the results are discussed and commented upon.

2. The mean flow

In ordinary plane Poiseuille flow the mean flow $U(y)$ in the streamwise direction (x) is given by $U(y) = 1 - y^2$ where y is the wall normal coordinate satisfying $-1 \leqslant y \leqslant 1$. One way of modelling helium II is to consider the superposition of a normal fluid and a superfluid where the two fluids interact via mutual friction. The mutual friction is then represented by a forcing term in the equations governing the flow. The mutual forcing term will then affect the normal fluid and thereby modify $U(y)$. Here, the study of helium II will thus be conducted with a model where a modified mean flow is considered. The general structure of the Navier–Stokes equation for a steady normal fluid mean flow, $U_n(y)$, in the streamwise (x) direction then becomes

$$-\frac{1}{R}U_n'' = -\frac{\mathrm{d}P}{\mathrm{d}x} + F_{MF}, \tag{2.1}$$

where U_n satisfies the boundary conditions

$$U_n(\pm 1) = 0. \tag{2.2}$$

In (2.1), R is the Reynolds number based on the centreline velocity and the channel half-width, a prime denotes differentiation with respect to y, the term $\mathrm{d}P/\mathrm{d}x$ is a uniform pressure gradient (given by $\mathrm{d}P/\mathrm{d}x = -2/R$) and the term F_{MF} represents the mutual forcing between the normal fluid and the superfluid. The subsequent modelling of the mutual forcing F_{MF} will essentially follow the steps presented by Godfrey *et al.* (2001). The superfluid has a uniform mean velocity profile, here denoted by U_s. The magnitude of U_s is chosen so that the mean flow rate of the superfluid becomes the same as for the Poiseuille profile $U(y) = 1 - y^2$. This implies that $U_s = 2/3$. In a number of studies, the mutual forcing F_{MF} has been found to be proportional to the difference between the normal fluid velocity and the superfluid velocity, (cf. Barenghi, Donnelly & Vinen 1983; Donnelly 1991). Therefore, the mutual forcing is zero in positions where the normal fluid mean velocity and the superfluid velocity have the same magnitude. This occurs thus in the positions where the Poiseuille profile $U(y)$ has the magnitude 2/3, i.e. for $y = \pm 1/\sqrt{3}$. Moreover, the superfluid vorticity is assumed to have a Gaussian distribution with peaks in the positions $y = \pm 1/\sqrt{3}$. This is motivated by the findings of Samuels (1992) who conducted simulations showing that the vortex filaments of the superfluid are concentrated to regions where the normal fluid and the superfluid have the same magnitudes. With these assumptions made, the mutual forcing F_{MF} can be modelled as

$$F_{MF} = N(y)(U_n(y) - U_s), \tag{2.3}$$

where $N(y)$ is given by

$$N(y) = F_m \left(\exp\left(-\frac{(y + 1/\sqrt{3})^2}{2\sigma^2} \right) + \exp\left(-\frac{(y - 1/\sqrt{3})^2}{2\sigma^2} \right) \right). \tag{2.4}$$

In (2.4), F_m is a parameter for controlling the strength of the mutual forcing term and σ is the standard deviation of the superfluid vorticity distribution. The magnitude of σ is assumed to be small in order to mimic the physical behaviour of a vorticity distribution concentrated around $y = \pm 1/\sqrt{3}$. The derived model (2.3), (2.4) for the mutual forcing F_{MF} is, of course, a highly simplified model which cannot be expected to mimic the full physics of helium II. Therefore, it is considered as a first step to studying how the transient growth of disturbances is affected by the presence of a forcing term. Continuing work could include a more sophisticated model including for example a feedback from the normal fluid to the superfluid. From (1.1), (2.3) and (2.4), it is clear that F_m must be proportional to the ratio of the superfluid density to the total density. Since the density ratio is strongly temperature-dependent, F_m is a strongly temperature-dependent parameter. A certain value of F_m represents therefore a certain temperature. Changing F_m will imply that another temperature is considered.

The development of the normal mean flow U_n is then described by the following equation and boundary conditions

$$-\frac{1}{R} U_n'' - N(y) U_n = -N(y)\frac{2}{3} + \frac{2}{R}, \tag{2.5}$$

$$U_n(\pm 1) = 0. \tag{2.6}$$

With $F_m = 0$, (2.5) gives merely the ordinary plane Poiseuille flow profile $U(y) = 1 - y^2$. If $F_m \neq 0$, the mean flow becomes modified by the mutual forcing represented by $N(y)$ in (2.5). The mean flow equation (2.5) is solved numerically with the finite-element method. The software used to solve (2.5) is version 3.2 of the commercially available program Comsol Multiphysics. The so-called coefficient form interface of the software is used. In the coefficient form interface, it is possible to tailor arbitrary systems of differential equations with boundary conditions and initial conditions. Equation (2.5) and the corresponding boundary conditions are implemented in the

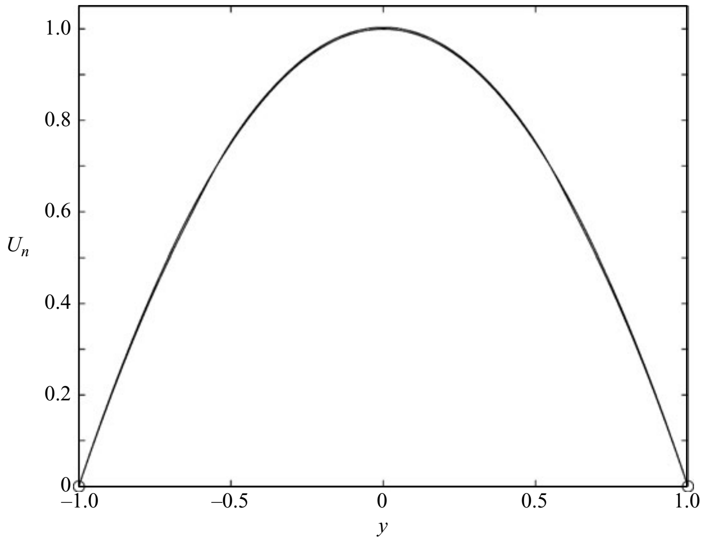

FIGURE 1. The mean flow U_n for $\sigma = 0.05$, $R = 1000$ with $F_m = 0$ and $F_m = 0.04$.

coefficient form interface of the software by setting the coefficients of a generic equation. The interval $-1 \leqslant y \leqslant 1$ is divided into 32 subintervals and on each subinterval a fourth-degree polynomial (a Lagrange quartic element) is used to approximate the solution. The type of program Comsol Multiphysics represents and the coefficient form interface have also been used in an earlier study of hydrodynamic stability where the program Femlab, i.e. the predecessor to Comsol Multiphysics, was used (see Bergström 2005).

If the parameters F_m and σ in (2.5) are chosen without any restrictions, it is possible to accomplish exotic forms of the mean flow U_n; for example, we can achieve huge magnitudes or even reverse the direction of the mean flow for certain combinations of F_m and σ. Therefore, in order to focus on physically realistic mean flows, F_m and σ are kept relatively small in this work. The parameters are chosen so that the mean flow does not deviate too much from the Poiseuille profile of an ordinary fluid. In fact, as will be shown in the next section, the mutual forcing will substantially change the transient growth characteristics, even if the parameters F_m and σ are small. In figure 1, the mean flow U_n obtained by the software is presented for $\sigma = 0.05$, $R = 1000$, $F_m = 0$ and $F_m = 0.04$. The mean flow profile is only slightly affected by the mutual forcing in this case and there is hardly a visible difference between the two curves. A closer examination shows that the maximum difference between the curves occurs for $y = 0$ where U_n of the $F_m = 0.04$ case has about 0.5 % higher magnitude than U_n of the $F_m = 0$ case. However, in figure 2 where the mean flow shear U_n' $(=(\mathrm{d}U_n/\mathrm{d}y))$ is presented for the same parameters as in figure 1, especially in the regions around $y = \pm 1/\sqrt{3}$, the $F_m = 0.04$ case clearly deviates from the $F_m = 0$ case owing to the mutual forcing. This is of particular interest since the shape of the mean shear is important for the transient growth. It is well known that a modification of the mean flow shear can alter the transient growth properties. An example for pipe flow is given in Bergström (2003*a*) where the mean flow modification is caused by the Coriolis force. Physically, the mean shear is important because the mean shear U_n' together with the wall normal velocity disturbance act as a forcing in the disturbance equations. This will be further investigated and elucidated in the next section.

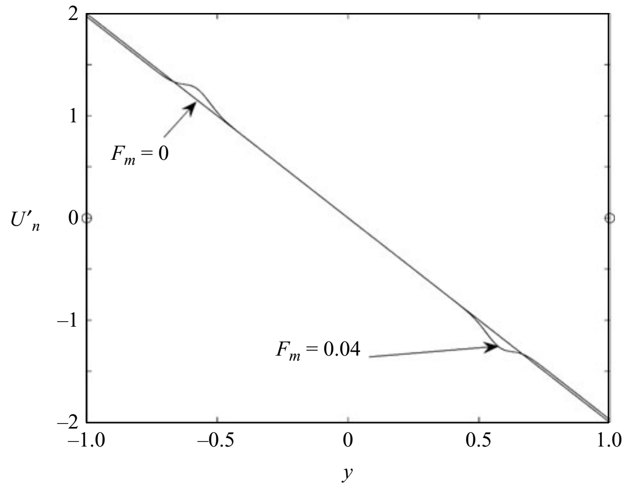

FIGURE 2. The mean shear U'_n for $\sigma = 0.05$, $R = 1000$ with $F_m = 0$ and $F_m = 0.04$.

3. The governing equations of the three-dimensional disturbance field

The mean flow U_n presented in § 2 will be exposed to a three-dimensional velocity disturbance field (u, v, w) and an associated pressure disturbance p. The disturbances u, v and w are thus the velocity disturbances in the streamwise (x), wall-normal (y) and spanwise (z) directions. The time development of small disturbances is governed by the linearized Navier–Stokes equations and the continuity equation. First, those equations are Fourier-transformed in the streamwise and spanwise directions, and α and β are introduced as the streamwise wavenumber and the spanwise wavenumber, respectively. This results in the following system of partial differential equations where u, v, w and p from now on represent the Fourier transformed forms of the disturbances,

$$\frac{\partial u}{\partial t} + \mathrm{i}\alpha U_n u + U'_n v + \mathrm{i}\alpha p + \frac{1}{R}(\alpha^2 + \beta^2)u - \frac{1}{R}u'' - N(y)u = 0, \tag{3.1a}$$

$$\frac{\partial v}{\partial t} + \mathrm{i}\alpha U_n v + \frac{\partial p}{\partial y} + \frac{1}{R}(\alpha^2 + \beta^2)v - \frac{1}{R}v'' - N(y)v = 0, \tag{3.1b}$$

$$\frac{\partial w}{\partial t} + \mathrm{i}\alpha U_n w + \mathrm{i}\beta p + \frac{1}{R}(\alpha^2 + \beta^2)w - \frac{1}{R}w'' - N(y)w = 0, \tag{3.1c}$$

$$\mathrm{i}\alpha u + v' + \mathrm{i}\beta w = 0, \tag{3.1d}$$

where $\mathrm{i} = \sqrt{-1}$. The boundary conditions for u, v and w are given by

$$u(\pm 1) = v(\pm 1) = w(\pm 1) = 0. \tag{3.2}$$

The boundary conditions for w together with (3.1c) mean that p must satisfy the boundary conditions,

$$p(\pm 1) = \frac{-\mathrm{i}}{\beta R} w''(\pm 1) \quad (\beta \neq 0). \tag{3.3}$$

In (3.1), the difference compared to the ordinary plane-Poiseuille-flow case is that the mean flow U_n and the mean shear U'_n are not the same as in the ordinary case and that the $N(y)$-term multiplied with u, v and w occurs. The mutual forcing has thus the potential to affect the disturbance development in two ways, through the mean flow terms U_n and U'_n and directly by $N(y)$ in the (3.1a–c). System (3.1) is solved numerically with the finite-element program described in connection to the mean flow equation in §2. For practical reasons, (3.1) and the mean flow equation (2.5) are solved simultaneously as a system of five equations which are all set up and defined in the coefficient form interface of the software. The initial condition for v, denoted v_0, will be the analytically accessible first symmetric eigenfunction of the $\alpha = 0$ form of v from the ordinary plane-Poiseuille-flow case, i.e.

$$v_0 = \cosh(\beta)\cos(\lambda_1 y) - \cos(\lambda_1)\cosh(\beta y), \tag{3.4}$$

where λ_1 is the first eigenvalue satisfying the relation

$$\lambda \tan(\lambda) + \beta \tanh(\beta) = 0, \tag{3.5}$$

(cf. Gustavsson 1991). The reason for choosing the symmetric v as an initial condition is that a symmetric wall-normal disturbance v induces an antisymmetric streamwise disturbance u through the forcing term $U_n' v$ in (3.1a). In earlier works, it has been found that the antisymmetric streamwise disturbance u exhibits the largest transient growth (see e.g. Gustavsson 1991; Butler & Farrell 1992). The initial streamwise disturbance denoted u_0 is chosen to be zero. In this way, the forcing of u by $U_n' v$ in (3.1a) can be exclusively studied. Through the continuity equation, the initial form of the spanwise disturbance w is then given by $w_0 = \mathrm{i}v_0'/\beta$, $(\beta \neq 0)$. A common way to characterize the time development of the disturbances is to consider the amplification of the energy density in Fourier space defined as

$$E_{Amp} = \frac{\displaystyle\int_{-1}^{1} (uu^* + vv^* + ww^*)\,\mathrm{d}y}{\displaystyle\int_{-1}^{1} (u_0u_0^* + v_0v_0^* + w_0w_0^*)\,\mathrm{d}y}, \tag{3.6}$$

where an asterisk denotes the complex conjugate. When the solutions for u, v and w have been obtained, a post-processing routine of the software is used to solve the integrals in (3.6) for each time step. The program Matlab is then used for the graphical presentation of the results.

4. Results

4.1. *The ordinary case $F_m = 0$*

First, the ordinary plane-Poiseuille-flow case, i.e. $F_m = 0$ in (2.4), will be considered in order to verify the reliability of the numerical software. The ordinary case has been thoroughly studied and analytical solutions are also available for the special case of streamwise independent disturbances, i.e. $\alpha = 0$. In the $\alpha = 0$ case, with the initial condition (3.4), the largest transient growth has been found to occur for $\beta = 1.98$ (cf. Gustavsson 1991). For $R = 1000$, $F_m = 0$, $\alpha = 0$ and $\beta = 1.98$, the peak value of E_{Amp} is slightly lower than 180 and occurs at approximately $t = 84$ (i.e. $t/R = 0.084$). This coincides well with the analytically accessible results for transient growth in ordinary plane Poiseuille flow presented by Gustavsson (1991) for the same values of the parameters. Although the largest transient growth in ordinary plane Poiseuille flow occurs for β close to two, it will be shown subsequently that this is not the case in the helium II case. However, since the present paper focuses on how the transient growth properties of a helium II fluid differ from the ordinary fluid case, $\beta = 1.98$ together with different values of F_m, σ and α will be used in most of the cases to be studied here. The value of β will thus be close to the value where ordinary plane

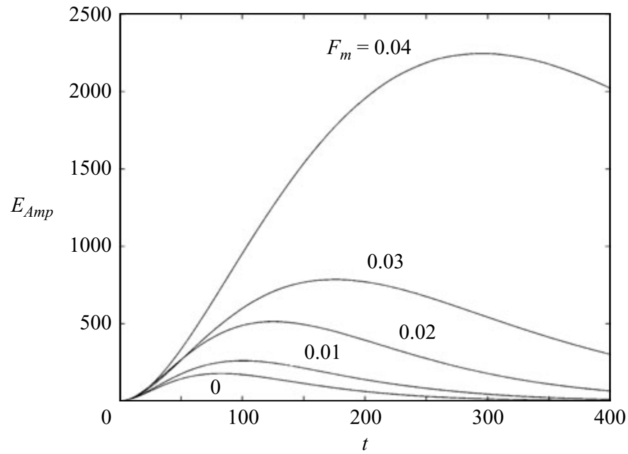

FIGURE 3. The time development of E_{Amp} for $\sigma = 0.05$, $R = 1000$, $\alpha = 0$, $\beta = 1.98$ and $F_m = 0$, 0.01, 0.02, 0.03, 0.04.

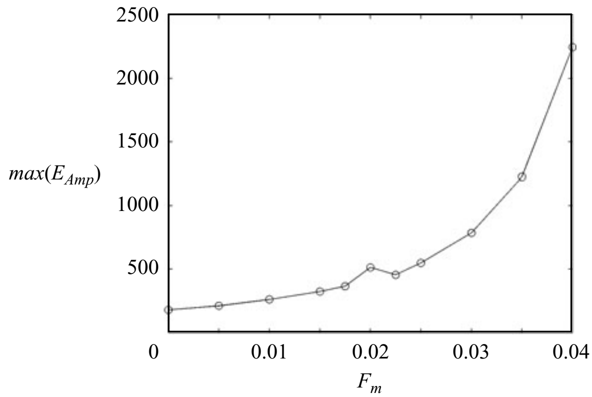

FIGURE 4. The peak value of E_{Amp} versus F_m for $\sigma = 0.05$, $R = 1000$, $\alpha = 0$ and $\beta = 1.98$.

Poiseuille flow exhibits the largest transient growth for the initial disturbance (3.4) and the results obtained for helium II can be directly compared to the results obtained for the corresponding wavenumber combination in the ordinary fluid case.

4.2. *The disturbance development for different magnitudes of F_m*

The development of E_{Amp} for different values of the parameter F_m will now be considered. Since F_m is strongly temperature dependent, varying F_m will represent cases of different temperature. In figure 3, the development of E_{Amp} is presented for $R = 1000$, $\alpha = 0$, $\beta = 1.98$, $\sigma = 0.05$ and $F_m = 0, 0.01, 0.02, 0.03, 0.04$. There is a strong increase of the peak amplification of E_{Amp} when F_m increases from 0 to 0.04. For $F_m = 0$, the peak value of E_{Amp} is slightly less than 180 and for $F_m = 0.04$, the peak value of E_{Amp} is 2245. The peak position of E_{Amp} also occurs substantially later as F_m increases. This implies that the $N(y)$-terms in (3.1) reduce the damping of the disturbances. When the magnitude of F_m is high enough, exponential growth occurs. In this case, exponential growth occurs for $F_m \simeq 0.06$. In figure 4, the peak amplification of E_{Amp} versus F_m is presented. For $F_m = 0.02$, the peak value of E_{Amp} deviates slightly from the overall pattern and a dent in the curve can be discerned. This

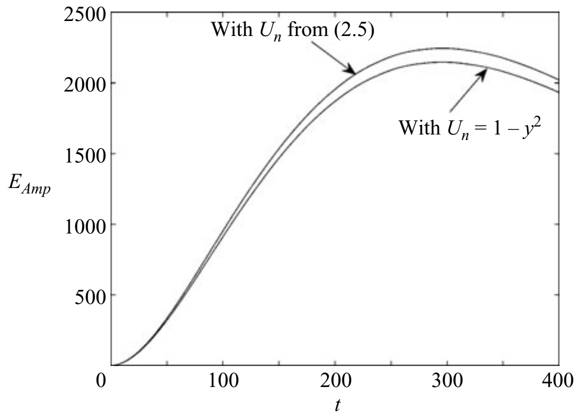

FIGURE 5. The time development of E_{Amp} for $\sigma = 0.05$, $R = 1000$, $\alpha = 0$, $\beta = 1.98$ and $F_m = 0.04$. The two curves represent the disturbance development due to the mean flow U_n from equation (2.5) and the mean flow $U_n \equiv U = 1 - y^2$.

deviation is caused by the mean shear and will be explained below. Since the mean shear U_n' is known to be crucial for the occurrence of transient growth, it is of interest to investigate to what extent U_n' is responsible for the increased transient growth observed in figure 3 when F_m increases. In order to investigate that, the mean flow $U(y) = 1 - y^2$ from ordinary plane Poiseuille flow will temporarily be used in (3.1). In figure 5, for the proper U_n from (2.5) and for $U_n \equiv U = 1 - y^2$, the development of E_{Amp} is presented for $R = 1000$, $\alpha = 0$, $\beta = 1.98$, $\sigma = 0.05$ and $F_m = 0.04$. With the proper U_n from (2.5), the peak value is 2245 whereas for $U_n \equiv U = 1 - y^2$, the peak value of E_{Amp} is 2147. The transient growth has thus been only slightly reduced compared to the case where the proper U_n and U_n' are used. This shows that the $N(y)$ terms in the disturbance equations (3.1) to a great extent cause the increased transient growth by reducing the damping of the disturbances. In the cases considered in figure 3, the modification of the mean flow and the mean shear is of minor importance for the increased transient growth. Nevertheless, the $U_n'v$ term in (3.1a) is necessary for any transient growth to occur. In fact, with the $U_n'v$ term in (3.1a) temporarily set to zero, the energy density E_{Amp} merely decay exponentially. The fact that the forcing is generic for the transient growth to occur concerns all types of parallel shear flows. In figure 4, for $F_m = 0.02$ a slight deviation from the overall pattern can be discerned. The extra boost in amplification indicated for $F_m = 0.02$ is due to a relatively large mean shear modification that happens to occur for the particular combination of the parameters $F_m$and σ. In figure 6, the mean shear U_n' is presented for $F_m = 0.015, 0.02$ and 0.025 with $\sigma = 0.05$. The $F_m = 0.02$ case clearly deviates from the other two cases which almost coincide. This deviation thus increases the relative influence of the mean shear U_n' on the transient growth in the $F_m = 0.02$ case compared to the $F_m = 0.015$ and 0.025 cases. This results in a little extra boost of the peak amplification of E_{Amp} and explains the dent in the curve observed in figure 4.

4.3. *Non-zero streamwise wavenumber*

The results presented so far show that the presence of the mutual friction forcing substantially increases the transient amplification of $\alpha = 0$ disturbances compared to the ordinary fluid case. The disturbance development for non-zero streamwise wavenumbers α will now be considered. In figure 7, the development of E_{Amp} is

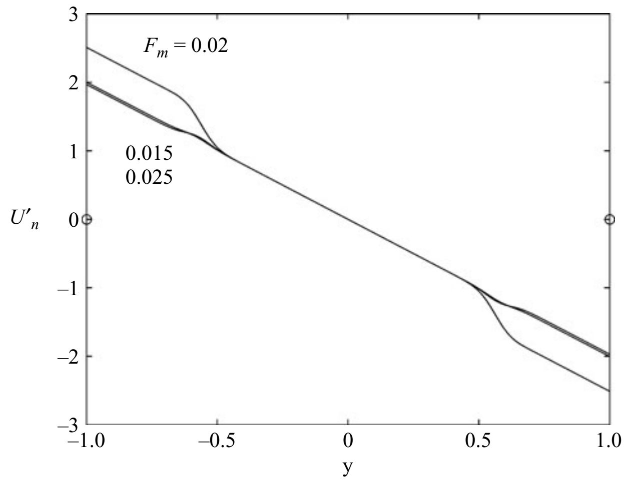

FIGURE 6. The mean shear U_n' for $\sigma = 0.05$, $R = 1000$ with $F_m = 0.015, 0.02$ and 0.025.

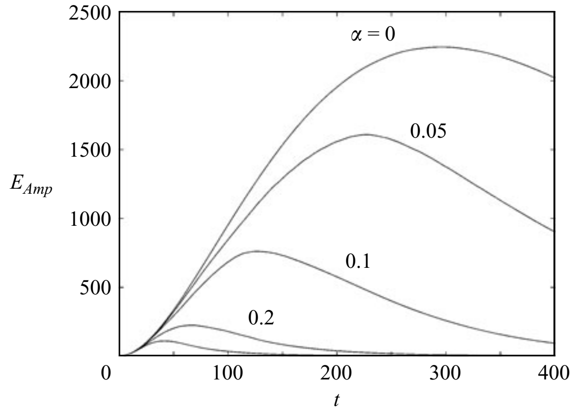

FIGURE 7. The time development of E_{Amp} for $\sigma = 0.05$, $R = 1000$, $\beta = 1.98$, $F_m = 0.04$ and $\alpha = 0$, 0.05, 0.1, 0.2 and 0.3.

presented for $R = 1000$, $F_m = 0.04$, $\sigma = 0.05$, $\beta = 1.98$ and the streamwise wavenumbers $\alpha = 0, 0.05, 0.1, 0.2$ and 0.3. When α increases, the peak amplification level quickly becomes reduced and the peak position of E_{Amp} occurs earlier. The peak amplification of E_{Amp} has decreased from 2245 for $\alpha = 0$ to 761 for $\alpha = 0.1$. This shows that as in the ordinary fluid case, streamwise elongated structures are the most transiently amplified ones in the helium II case. Other combinations of F_m and σ give the same result, i.e. the amplification decreases as α increases. The fact that elongated structures are favoured by the transient growth is a common property of parallel shear flows. Since $\alpha = 0$ exhibits the largest transient amplification in the helium II case also, $\alpha = 0$ will be used in the subsequent investigations.

4.4. *The effect of the spread of the vorticity distribution on the amplification level*

The magnitude of the standard deviation σ of the superfluid vorticity distribution affects the mutual forcing by regulating the spread of the vortices. If σ increases, the vorticity distribution becomes wider and has thereby the potential to affect the flow to a greater extent. In order to investigate how the width of the vorticity distribution

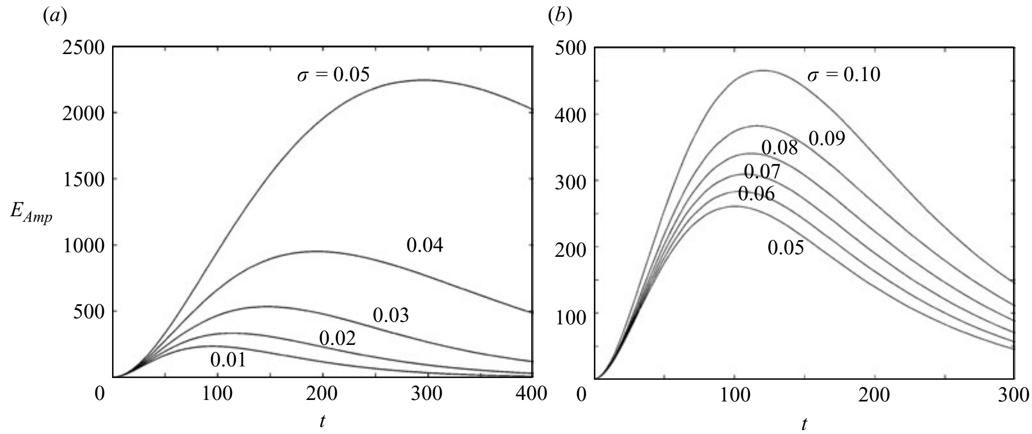

FIGURE 8. The time development of E_{Amp} for (*a*) $F_m = 0.04$, $R = 1000$, $\alpha = 0$, $\beta = 1.98$ and $\sigma = 0.01$, 0.02, 0.03, 0.04 and 0.05; (*b*) $F_m = 0.01$, $R = 1000$, $\alpha = 0$, $\beta = 1.98$ and $\sigma = 0.05$, 0.06, 0.07, 0.08, 0.09 and 0.10.

affects the disturbance development, σ will be varied for constant magnitudes of F_m. Since F_m is strongly temperature dependent, this will show how the disturbance development varies with σ at a certain temperature. The influence of the standard deviation on the transient growth is presented in figure 8(*a*) where the development of E_{Amp} is shown for $F_m = 0.04$, $R = 1000$, $\alpha = 0$, $\beta = 1.98$ and $\sigma = 0.01, 0.02, 0.03, 0.04$ and 0.05. When σ is reduced from 0.05 to 0.04, the peak amplification of E_{Amp} is reduced from 2245 to 951, i.e. by almost 58 %. The peak position of E_{Amp} occurs substantially earlier and moves from $t = 296$ to $t = 194.5$. When σ is reduced further, the peak amplification continues to decrease and the position of the peak occurs earlier. Finally, the amplification of the ordinary plane Poiseuille flow is approached when σ approaches zero. With $F_m = 0.04$, exponential growth occurs if σ is increased above approximately 0.065. In figure 8(*b*) the development of E_{Amp} is presented for $R = 1000$, $F_m = 0.01$, $\alpha = 0$, $\beta = 1.98$ and $\sigma = 0.05, 0.06, 0.07, 0.08, 0.09$ and 0.1. The peak amplification decreases when σ decreases, but not to the same extent as in the previous case where F_m was larger and σ smaller. By halving σ from 0.1 to 0.05, the peak amplification decreases from 466 to 261. The reduction of the amplification as σ decreases is thus not as pronounced as it was for smaller σ in the $F_m = 0.04$ case in figure 8(*a*). Compared to the previous case, the position of the peak value of E_{Amp} occurs somewhat later when σ increases. With $F_m = 0.01$, exponential growth occurs for σ above approximately 0.48, i.e. for a substantially larger value of σ than in the $F_m = 0.04$ case.

4.5. *The spanwise wavenumber* β

In general, a spanwise dependence (i.e. $\beta \neq 0$) is crucial for substantial transient growth to occur in planar shear flows. Although transient growth can also occur in the two-dimensional case (see e.g. Farrell 1988), the amplification levels in the two-dimensional case are far from the amplification levels found in the three-dimensional case. As mentioned earlier, for ordinary plane Poiseuille flow, the largest transient growth occurs for the spanwise wavenumber β close to 2. However, for helium II, this is in general not the case. This is evident in figure 9 where the time development of E_{Amp} is presented for $F_m = 0.04$, $\sigma = 0.05$, $R = 1000$, $\alpha = 0$ and the spanwise wavenumbers $\beta = 1.8$, 1.9, 1.98, 2.1 and 2.2. Here, the transient growth increases

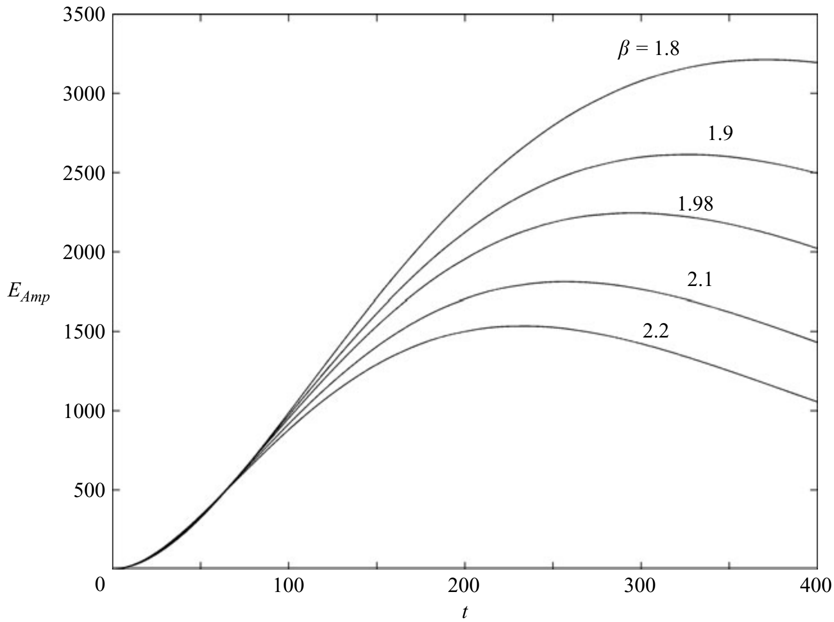

FIGURE 9. The time development of E_{Amp} for $F_m = 0.04$, $R = 1000$, $\alpha = 0$, $\sigma = 0.05$ and $\beta =$ 1.8, 1.9, 1.98, 2.1 and 2.2.

when β decreases. It continues to do so when β is further reduced below 1.8. This is strikingly different to the ordinary fluid case where the maximum amplification occurs for β close to 2. Obviously, the modification of the ordinary system by the $N(y)$-terms in (3.1) implies that the way the peak amplification depends on β becomes modified. The presence of the mutual forcing $N(y)$ in (3.1) thus changes the behaviour found in the ordinary case. An explanation for this result could be that the relative influence of the $N(y)$ terms that support growth in (3.1) increases when β decreases. This is further developed and commented upon in connection with (5.1) in § 5. However, the exact way in which the peak amplification level depends on β is not easily revealed, not even in the ordinary case. In (3.1*a*–*d*), β occurs explicitly in several terms as well as implicitly in u, v, w and p. Also, β is included in the initial conditions. When the magnitude of F_m is reduced enough, a β-behaviour like that in the ordinary case is found.

4.6. *The Reynolds-number dependence of the peak amplification*

It is well known that in parallel shear flows of an ordinary fluid, the peak energy density amplification is proportional to the square of the Reynolds number (see e.g. Butler & Farrell 1992). The structure of the equations governing helium II alters the way the peak amplification of E_{Amp} depends on the Reynolds number. This is evident in figure 10 where the peak value of E_{Amp} versus R is presented for $F_m = 0.01$, $\sigma = 0.05$, $\alpha = 0$ and $\beta = 1.98$. The curve represents a quadratic R-dependence normalized to the $R = 1000$ case and the stars represent the actual peak amplifications for different values of R. The peak amplification in the helium II case does not exhibit a quadratic Reynolds-number dependence, instead a stronger Reynolds-number dependence is indicated in figure 10. This can be explained by

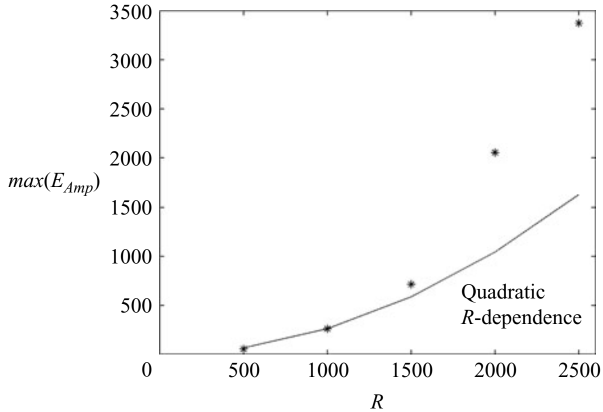

FIGURE 10. The peak value of E_{Amp} versus the Reynolds number R. The stars represent the actual results and the curve represents a quadratic R-dependence.

introducing the scalings (cf. Gustavsson 1991; Schmid & Henningson 2001)

$$\bar{u} = \frac{u}{\beta R}, \quad \bar{t} = \frac{t}{R}, \quad \bar{v} = v. \tag{4.1}$$

The largest peak amplification occurs for $\alpha = 0$ disturbances. By inserting the transformations (4.1) into the $\alpha = 0$ version of (3.1a) the following modified form of (3.1a) is obtained:

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \frac{U_n' \bar{v}}{\beta} + \beta^2 \bar{u} - \bar{u}'' - RN(y)\bar{u} = 0. \tag{4.2}$$

If $N(y) = 0$, $\bar{u}$ will not depend on the Reynolds number R and the energy density associated with u is given by $\beta^2 R^2 \int_{-1}^{1} \bar{u}^2 \mathrm{d}y$; that is, the energy density associated with u depends on R^2 since $\bar{u}$ is independent of R. If $N(y) \neq 0$ in (4.2), $\bar{u}$ will depend on R and the R^2 dependence of the energy density associated with u becomes thereby altered. Thus, the R dependence found for transient growth of disturbances in an ordinary fluid does not hold any longer. This will also be substantialized with a model equation in the Appendix. In general, the R^2 dependence of the peak energy density amplification of disturbances found in the ordinary fluid case does not hold if the fluid properties are modified. For example, in the study of transient growth of disturbances in a Jeffrey fluid flowing through a circular pipe, Bergström (2003b) showed that the R^2 dependence of the energy density peak amplification is not valid.

5. Discussion

The transient growth of small disturbances in plane Poiseuille flow of a helium II fluid has been investigated. The results show that the transient growth properties are substantially modified compared to the case of an ordinary fluid. The mutual forcing boosts the transient growth by counteracting the terms that restrain amplification. This is the major reason for the increased amplification compared to the ordinary case. However, there is also an additional less significant affect on the growth caused by the modification of the mean flow. For particular combinations of F_m and σ, the mean flow modification can, however, play a more significant role. For example, the dent observed in figure 4 for $F_m = 0.02$ is caused by the mean flow modification which was found to be larger for $F_m = 0.02$ than for F_m slightly lower or slightly higher than 0.02. However, we must bear in mind that the dent occurs as a result of a relatively large mean shear modification for a particular combination of the parameters in the present model. The physical relevance of the dent should therefore be questioned as long as it has not been verified by a more sophisticated model.

The increased transient growth occurs because the damping of the disturbances decreases owing to $N(y)$. This can be illustrated by considering the change of the energy density of the velocity disturbances. For example, in the $\alpha = 0$ case, the change of the energy density (E_u) associated with the streamwise disturbance velocity u can be expressed as

$$\frac{\partial E_u}{\partial t} = -\int_{-1}^{1} U_n' v u \, \mathrm{d}y - \frac{1}{R}\int_{-1}^{1} (\beta^2 u^2 + u'^2) \, \mathrm{d}y + \int_{-1}^{1} N(y) u^2 \, \mathrm{d}y. \qquad (5.1)$$

This expression is obtained by multiplying (3.1a) by u and integrating over the channel height and exploiting the boundary conditions. When $N(y) = 0$, the only term that can increase the energy density is the $U_n' v u$ term in (5.1). This term causes the transient growth of the streamwise disturbance. When $N(y) \neq 0$, the $N(y)$-term which is always positive counteracts the always negative $1/R$-terms in (5.1) and thereby reduces the damping. When $N(y)$ becomes sufficiently dominant, exponential growth may occur.

Concerning the Reynolds-number dependence of the peak amplification, in figure 10 it is evident that the R^2-dependence of the peak amplification of the energy density found for an ordinary fluid does not hold in the helium II case. Because of the $N(y)$ term, the well-known R^2-dependence becomes modified and the peak amplification of the energy density depends more strongly on R in the helium II case than in the ordinary fluid case.

In this work, a single $\alpha = 0$ eigenmode from the ordinary fluid case is used as an initial disturbance. For $\alpha = 0$ disturbances of an ordinary fluid, the initial disturbance (3.4) gives a disturbance development close to the optimal one, i.e. the disturbance giving the largest possible amplification. However, the initial disturbance of an optimal disturbance is, in general, composed of several eigenmodes. Thus, although the transient amplifications found in this work are substantially larger than in the case of an ordinary fluid, they are not necessarily the largest possible amplifications that can be obtained in the helium II case.

The present investigation concerns the time development of small three-dimensional disturbances, i.e. the linearized initial-value problem has been considered and nonlinear effects are not taken into account. The intrinsic mechanism of transient growth is, however, a completely linear mechanism associated with the non-modal properties of the equations governing the disturbances in parallel shear flows. An alternative more physical way of describing the core of the transient growth is that the $U_n' v$ term in (3.1a) forces the streamwise velocity disturbance. Although transient growth is caused by a linear mechanism, nonlinearity would affect the transient growth and a continuation of the work could be to investigate how nonlinearity would modify the disturbance development found in the present work. Another continuation of the work presented here could be to invoke the disturbance field of the superfluid. The superfluid disturbances will then be affected by a forcing from the normal fluid. Such an approach will also modify the mutual forcing terms. Alternative models for modelling the mutual forcing can also be considered.

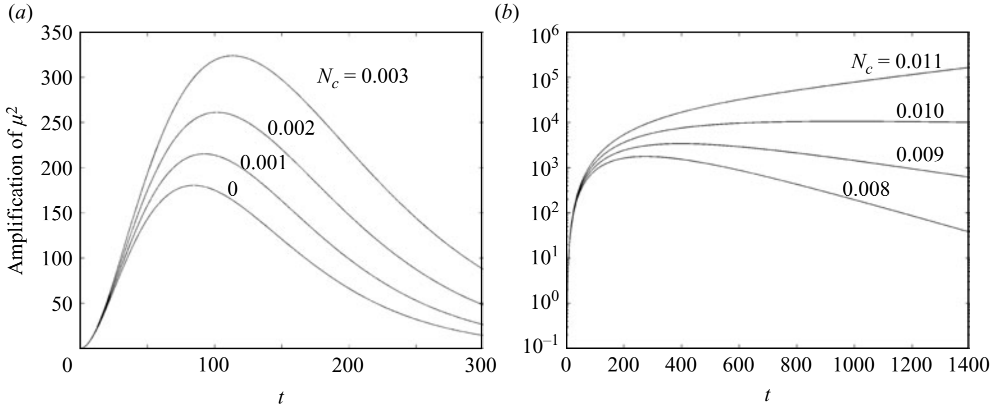

FIGURE 11. Development of μ^2 from the simplified model equation (A 5) for (*a*) the parameter $N_c = 0$, 0.001, 0.002 and 0.003; (*b*) $N_c = 0.008$, 0.009, 0.010 and 0.011.

Appendix. A simplified model equation

A simplified model equation that to some extent mimics the specific behaviour of helium II found in the present work will be derived. For $\alpha = 0$, equation (3.1*a*) and the corresponding boundary conditions become

$$\frac{\partial u}{\partial t} + \frac{1}{R}(\beta^2 - RN(y))u - \frac{1}{R}u'' = -U_n' v, \qquad \text{(A 1)}$$

$$u(y = \pm 1) = 0. \qquad \text{(A 2)}$$

By inserting the ansatz $u \sim \hat{u}(y)e^{-ct}$ into (A 1) and considering the homogeneous part of the resulting equation, the following eigenvalue problem is obtained:

$$\hat{u}'' + (Rc - \beta^2 + RN(y))\hat{u} = 0, \qquad \text{(A 3)}$$

$$\hat{u}(\pm 1) = 0. \qquad \text{(A 4)}$$

In a simplified model where $N(y)$ is replaced by a constant N_c, the solution to the eigenvalue problem (A 3), (A 4) becomes $\hat{u} = \sin(n\pi y)$ and $c = (n^2\pi^2 + \beta^2 - RN_c)/R$ where $n = 1, 2, 3, \ldots$.

A simple model equation that mimics the transient time development found in this work can then have the form

$$\frac{d\mu}{dt} + \left(\frac{\pi^2 + \beta^2 - RN_c}{R}\right)\mu = \upsilon \exp\left(-\frac{\lambda^2 + \beta^2 - RN_c}{R}t\right), \qquad \text{(A 5)}$$

$$\mu(0) = 0. \qquad \text{(A 6)}$$

In (A 5), μ is a transiently amplified quantity where the term $(\pi^2 + \beta^2 - RN_c)/R$ represents the damping of μ and where the term $(\lambda^2 + \beta^2 - RN_c)/R$ represents the damping of a quantity forcing μ. The forcing quantity $\upsilon \exp(-(\lambda^2 + \beta^2 - RN_c)t/R)$ on the right-hand side of (A 5) could thus represent the $U_n' v$ term in (3.1*a*). The parameters λ, β, R, υ and N_c are all constants. The solution of (A 5) is readily given by

$$\mu(t) = \frac{\upsilon R}{\pi^2 - \lambda^2}\left(\exp\left(-\frac{\lambda^2 + \beta^2 - RN_c}{R}t\right) - \exp\left(-\frac{\pi^2 + \beta^2 - RN_c}{R}t\right)\right). \qquad \text{(A 7)}$$

In (A 7), it is easily verified that an increase of N_c decreases the damping in both exponential terms. In figure 11(*a*), the time development of μ^2 is considered for

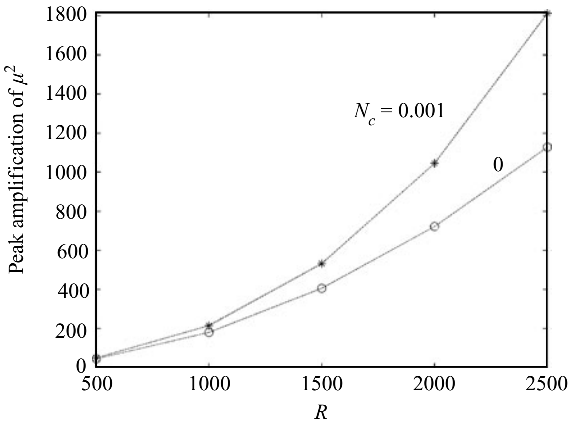

FIGURE 12. The peak amplification of μ^2 from the simplified model equation (A5) versus the parameter R for $N_c = 0$ and $N_c = 0.001$.

$\lambda = 2.4879$ (i.e. the same value as λ_1 from (3.5)), $\beta = 1.98$, $R = 1000$, $\upsilon = 0.4347$ and $N_c = 0$, 0.001, 0.002 and 0.003. The value of υ is chosen to obtain a peak amplification of approximately 180 for $N_c = 0$ (i.e. the same amplification as in the real case). The transient amplification of μ^2 increases as N_c increases. If N_c becomes large enough, exponential growth will occur. This behaviour is thus analogous to the real case. With the above presented values of the parameters, exponential growth occurs for N_c just above 0.010 since RN_c then becomes larger than $\lambda^2 + \beta^2$ in (A 7). This is shown in figure 11(*b*) where the time development of μ^2 is presented on a logarithmic scale for larger values of N_c. For $N_c = 0.011$, exponential growth occurs as expected. In (A 7), it is also obvious that an increase of the Reynolds number will make the N_c term more dominant. In figure 12, the peak amplification of μ^2 versus R is presented for $N_c = 0$ and $N_c = 0.001$. For $N_c = 0$, the peak value of μ^2 depends on the square of the Reynolds number whereas for $N_c = 0.001$ the peak value of μ^2 does not depend on R^2. Instead a stronger R-dependence is indicated.

REFERENCES

BARENGHI, C. F., DONNELLY, R. J. & VINEN, W. F. 1983 Friction on quantized vortices in helium II. A review. *J. Low Temp. Phys.* **52**, 189.

BARENGHI, C. F., SAMUELS, D. C., BAUER, G. H. & DONNELLY, R. J. 1997 Superfluid vortex lines in a model of turbulent flow. *Phys. Fluids* **9**, 2631.

BERGSTRÖM, L. 1993 Optimal growth of small disturbances in pipe Poiseuille flow. *Phys. Fluids* A **5**, 2710–2720.

BERGSTRÖM, L. 1995 Transient properties of a developing laminar disturbance in pipe Poiseuille flow. *Eur. J. Mech.* B/*Fluids* **14**, 601.

BERGSTRÖM, L. B. 2003*a* The effect of the Earth's rotation on the transient amplification of disturbances in pipe flow. *Phys. Fluids* **15**, 3028–3035.

BERGSTRÖM, L. B. 2003*b* Transient growth of small disturbances in a Jeffrey fluid flowing through a pipe. *Fluid Dyn. Res.* **32**, 29–44.

BERGSTRÖM, L. B. 2005 Nonmodal growth of three-dimensional disturbances on plane Couette–Poiseuille flows. *Phys. Fluids* **17**, 014105.

BUTLER, K. M. & FARRELL, B. F. 1992 Three-dimensional optimal perturbations in viscous shear flow. *Phys. Fluids* A **4** (8), 1637.

DAVIES, S. & WHITE, C. M. 1928 An experimental study of the flow of water in pipes of rectangular section. *Proc. R. Soc. Lond.* A 119, 92–107.

Donnelly, R. J. 1991 Quantized vortices in helium II. Cambridge University Press.

Farrell, B. F. 1988 Optimal excitation of perturbations in viscous shear flow. *Phys. Fluids* **31**(8), 2093–2102.

Godfrey, S. P., Samuels, D. C. & Barenghi, C. F. 2001 Linear stability of laminar plane Poiseuille flow of helium II under a non-uniform mutual friction forcing. *Phys. Fluids* **13**(4), 983–990.

Gustavsson, L. H. 1991 Energy growth of three-dimensional disturbances in plane Poiseuille flow. *J. Fluid Mech.* **224**, 241–260.

Heisenberg, W. 1924 Uber stabilität und turbulenz von flussigkeitsströmen. *Annln Phys., Lpz.* (4) **74**, 577–627.

Herbert, T. 1976 Periodic secondary motions in a plane channel. *Proc. 5th Intl Conf. Numer. Methods Fluid Dyn.* (ed. A. I. van de Vooren & P. J. Zandbergen), pp. 235–240. Springer.

Kaskel, A. 1961 Experimental study of the stability of pipe flow. II. Development of disturbance generator. Jet Propulsion Laboratory, California Institute of Technology, *Tech. Rep.* 32–138.

Klingmann, B. G. 1992 On transition due to three-dimensional disturbances in plane Poiseuille flow. *J. Fluid Mech.* **240**, 167–195.

Landahl, M. T. 1975 Wave breakdown and turbulence. *SIAM J. Appl. Maths* **28**, 735–756.

Landahl, M. T. 1980 A note on an algebraic instability of inviscid parallel shear flows. *J. Fluid Mech.* **98**, 243–251.

Lin, C. C. 1945 On the stability of two-dimensional parallel flows. Parts I–III. *Q. Appl. Maths* **3**, 117–142, 218–234, 277–301.

Mayer, E. W. & Reshotko, E. 1997 Evidence for transient disturbance growth in a 1961 pipe-flow experiment. *Phys. Fluids* **9**, 242.

Melotte, D. J. & Barenghi, C. F. 1998 Transition to normal fluid turbulence in helium II. *Phys. Rev. Lett.* **80**, 4181.

Meskyn, D. & Stuart, J. T. 1951 Stability of viscous motion between parallel planes of finite disturbances. *Proc. R. Soc. Lond* A **208**, 517–526.

Morkovin, M. V. 1969 The many faces of transition. In *Viscous Drag Reduction* ed. C. S. Wells) Plenum.

Orszag, S. A. 1971 Accurate solution of the Orr–Sommerfeld stability equation. *J. Fluid Mech.* **50**, 689–703.

Pekeris, C. L. & Scholler, B. 1971 Stability of plane Poiseuille flow to periodic disturbances of finite amplitude. *Proc. Natl Acad. Sci. USA* **68**, 197–199, 1434–1435.

Reddy, S. C. & Henningson, D. S. 1993 Energy growth in viscous channel flows. *J. Fluid Mech.* **252**, 209–238.

Reynolds, W. C. & Potter, M. C. 1967 Finite-amplitude instability of parallel shear flows. *J. Fluid Mech.* **27**, 465–492.

Samuels, D. C. 1992 Velocity matching and Poiseuille pipe flow of superfluid helium. *Phys. Rev.* B **46**, 11714.

Schlichting, H. 1933 Berechnung der anfachung kleiner storungen bei der plattenstromung. *Z. Angew. Math. Mech.* **13**, 171–174.

Schmid, P. J. & Henningson, D. S. 1994 Optimal energy density growth in Hagen–Poiseuille flow. *J. Fluid Mech.* **277**, 197–225.

Schmid, P. J. & Henningson, D. S. 2001 *Stability and Transition in Shear Flows*. Springer.

Smith, M. R., Donnelly, R. J., Goldenfeld, N. & Vinen, W. F. 1993 Decay of vorticity in homogeneous turbulence. *Phys. Rev. Lett.* **71**, 2583.

Squire, H. B. 1933 On the stability of three-dimensional disturbances of viscous flow between parallel walls. *Proc. R. Soc. Lond* A **142**, 621–628.

Stuart, J. T. 1960 On the nonlinear mechanics of wave disturbances in stable and unstable parallel flows. Part 1. *J. Fluid Mech.* **9**, 353–370.

Tollmien, W. 1929 Uber die entstehung der turbulenz. *Nachr. Ges. Wiss. Göttingen Math.-phys K1*, 21–44.

Watson, J. 1960 On the nonlinear mechanics of wave disturbances in stable and unstable parallel flows. Part 2. *J. Fluid Mech.* **9**, 371–389.